Compitino di Matematica per la Fisica

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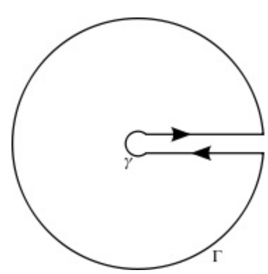
1. (a) First note that
$$\left(\frac{1-i}{\sqrt{2}}\right)^i = e^{\frac{\pi}{4}-2\pi n}$$
so the logarithm equals
$$\log\, i\left(\frac{1-i}{\sqrt{2}}\right)^i = \frac{\pi}{4} - 2\pi n + i(\frac{\pi}{2} + 2\pi m)$$
for $n, m \in \mathbb{Z}$.

   (b) Write
$$\log\,(1+i\sqrt{3}) = \log 2 + i\frac{\pi}{3} + 2\pi i m$$
so $r = \sqrt{(\log 2)^2 + (\pi/3 + 2\pi m)^2}$ and $\theta = \frac{\pi/3+2\pi m}{\log 2}$.

2. Write
$$2(1+i) = 2\sqrt{2}e^{i\frac{\pi}{4}+i2\pi m}$$
Taking the fifth root,
$$z = 2^{3/10}e^{i\frac{\pi}{20}+i\frac{2\pi m}{5}}$$

3. Use
$$\delta(g(x)) = \sum_i \frac{\delta(x-x_i)}{|g'(x)|}$$
where x_i are the zeros of $g(x)$. Then note that $x^2-6x+8 = (x-2)(x-4)$ so $x_i = (2,4)$. Plugging it in immediately gives $I = 42$.

4. First we write
$$\int_0^\infty \frac{dx}{\sqrt{x}(x+4)} = \lim_{\rho\to\infty,\epsilon\to 0^+}\int_\epsilon^\rho \frac{dx}{\sqrt{x}(x+4)}$$
and then we note that a integral along the closed loop C



equals,

$$\int_C \frac{dz}{\sqrt{z}(z+4)} = 2\pi i Res(f; z_0 = -4) = \pi \tag{1}$$

The contour can be split up into four pieces; two (with opposite direction) along the positive real axis and two additional contours γ and Γ with radii $\rho >> 1$ and $\epsilon << 1$. Writing $z = \rho\, e^{i\theta}$ for Γ we note

$$\int_\Gamma \frac{dz}{\sqrt{z}(z+2)} \leq \int_0^{2\pi} \frac{\rho}{\sqrt{\rho}(\rho+4)} d\theta \to 0$$

while for γ we write $z = \epsilon\, e^{i\theta}$,

$$\int_\gamma \frac{dz}{\sqrt{z}(z+2)} \geq \int_0^{2\pi} \frac{\epsilon}{\sqrt{\epsilon}(\epsilon+4)} d\theta \to 0$$

(we only have integration over θ since the circles have constant radii). Thus the only contributions to the LHS of (1) comes from the integration along the positive real axis. Just above the axes, $z = x$ while below $z = xe^{2\pi i}$ and the square root has a branch cut along all $x > 0$. Therefore,

$$\int_C \frac{dz}{\sqrt{z}(z+4)} = \int_\epsilon^\rho \frac{dx}{\sqrt{x}(x+4)} - \int_\rho^\epsilon \frac{dx}{\sqrt{x}e^{i\pi}(x+4)} = 2\int_\epsilon^\rho \frac{dx}{\sqrt{x}(x+4)}$$

This is of course just two times (1) so

$$\int_0^\infty \frac{dx}{\sqrt{x}(x+4)} = \frac{\pi}{2}$$

Alternative method:
There is in fact a much simpler way to evaluate the integral without invoking any sort of complex analysis: Introduce a new variable $y = \frac{1}{2}\sqrt{x}$ and evaluate the integral directly,

$$4\int_0^\infty \frac{dy}{4(y^2+1)} = \arctan y|_0^\infty = \frac{\pi}{2}$$

5. In this problem we are given $|z-1| > 1$ which describes an open set centered at $z_0 = 1$. We begin by writing

$$f(z) = \frac{z+1}{(z-3)(z-1)} = \frac{2}{z-3} - \frac{1}{z-1}$$

where the Laurent series expansion for the second term is trivial, its simply $-1/(z-1)$. The first term is slightly more tricky because the Laurent series expansion will look different depending on region for z. If we write,

$$1 < |z-1| < 2: \qquad \frac{2}{z-3} = -\frac{1}{1-\frac{z-1}{2}}, \qquad |\frac{z-1}{2}| < 1$$

$$|z-1| > 2: \qquad \frac{2}{z-3} = \frac{2}{z-1} \times \frac{1}{1-\frac{2}{z-1}}, \qquad |\frac{2}{z-1}| < 1$$

we find using the geometric series expansion,

$$1 < |z-1| < 2: \qquad f(z) = -\frac{1}{z-1} - \sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n$$

$$|z-1| > 2: \qquad f(z) = -\frac{1}{z-1} + \frac{2}{z-1}\sum_{n=0}^{\infty}\left(\frac{2}{z-1}\right)^n = -\frac{1}{z-1} + \sum_{n=0}^{\infty}\left(\frac{2}{z-1}\right)^{n+1}$$

Note: The point $|z-1| = 2$ is excluded since $z = 3$ is a pole.

6. (a) Since the integral scales like $1/x^2$ for large x we are allowed to use Jordan's lemma. Noting that the integral has simple poles at $z_0 = \pm 2i$ and $z_0 = \pm 4i$ but that only the plus one lies in the UHP, a quick calculation shows

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)(x^2+16)} dx = 2\pi i \left(Res(f; z_0 = 2i) + Res(f; z_0 = 4i) \right) = \frac{\pi}{6}$$

(b) Write $z = e^{i\theta}$ so $\cos\theta = \frac{1}{2}(z + 1/z)$ with $d\theta = dz/(iz)$, then

$$\int_0^{2\pi} \frac{d\theta}{2 - \cos\theta} = 2i \int_{|z|=1} \frac{dz}{z^2 - 4z + 1}$$

Thus we have simple poles at $z_0 = 2 \pm \sqrt{3}$ but only $2 - \sqrt{3}$ lies inside $|z| = 1$. A quick calculation gives,

$$2\pi i Res(f; z_0 = 2 - \sqrt{3}) = \frac{2\pi}{\sqrt{3}}$$